Midterm Test suggested solution

1. Let f be a continuous function defined on $[a,b]$ with $f(a)=f(b)$.

   (a) Suppose f' exists on (a,b). Show that there is $\xi \in (a,b)$ such that $f'(\xi)=0$.

   Proof. As f is continuous on $[a,b]$, f attains its maximum and minimum on $[a,b]$. That is to say $\exists\, p,q \in [a,b]$ such that

   $$f(p) \le f(x) \le f(q) \;\; \forall\, x \in [a,b].$$

   If $f(p)=f(q)$, f is a constant function. So $f' \equiv 0$.
   Suppose $f(p) < f(q)$, we may assume $f(q) > f(a) = f(b)$. Then q is a interior maximum. If $f'(q) > 0$, then $\exists \delta > 0$ such that

   $$\frac{f(q+h)-f(q)}{h} > 0 \;\; \forall h \in (-\delta,\delta) \setminus \{0\}$$

   which contradicts with the fact that f attains maximum at q. Similarly, $f'(q)$ cannot be negative. So $f'(q)=0$. □

   (b) If the continuity of f at a and b is removed, does the part (i) still hold?

   Proof. No. Choose $f : [0,1] \to \mathbb{R}$ where $f(0)=f(1)=0$ and $f(x)=x$ if $x \in (0,1)$. □

2. Let f and g be continuous functions on $[a,b]$. Suppose that f and g are differentiable on (a,b) with $|f'(x)| \le 1 \le |g'(x)|$ on (a,b). Show that $|f(x)-f(a)| \le |g(x)-g(a)|$ on $[a,b]$.

   Proof. Let $x \in (a,b]$, by mean value theorem we can find $c,d \in (a,b)$ such that

   $$f(x)-f(a) = f'(c)(x-a) \;\text{ and }\; g(x)-g(a) = g'(d)(x-a).$$

   Noted that c and d may not be the same. Then

   $$|f(x)-f(a)| = |f'(c)||x-a| \le |x-a| \le |g'(d)||x-a| = |g(x)-g(a)|.$$

   When $x=a$, the inequality trivially holds. □

3. Define $f : [-1,1] \to \mathbb{R}$ by $f(t)=-1$ if $t<0$ and $f(t)=1$ if $t \ge 0$. Let

   $$F(x) = \int_{-1}^{x} f(t)\, dt$$

   for $x \in (-1,1]$. If F differentiable on $(-1,1)$?

Proof. f is clearly integrable. So $F(a+b) = F(a) + \int_a^{a+b} f$ for a, b and $a+b \in [-1,1]$. For $h \in (-1,1)$,

$$\frac{F(h) - F(0)}{h} = \frac{1}{h} \cdot \int_0^h f(t)\, dt.$$

If $h > 0$, then

$$\frac{F(h) - F(0)}{h} = \frac{1}{h} \cdot \int_0^h 1\, dt = 1.$$

If $h < 0$,

$$\frac{F(h) - F(0)}{h} = \frac{1}{h} \cdot \int_0^h -1\, dt = -1.$$

So F is not differentiable at $x = 0$. By fundamental theorem of Calculus, F is differentiable on c where f is continuous. So F is differentiable on $(-1,1) \setminus \{0\}$. $\square$

4. If f is nonnegative Riemann integrable function on $[a,b]$, does it imply $\sqrt{f}$ Riemann integrable on $[a,b]$?

Proof. Yes. Since $f \in R[a,b]$, for any $\epsilon > 0$, there exists $\delta > 0$ such that whenever $\mathcal{P}$ is a partition with $||\mathcal{P}|| < \delta$,

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) < \epsilon^2.$$

For such $\mathcal{P}$, denote $(m_i)M_i = (\inf)\sup\{f(x) : x \in [x_i, x_{i+1}]\}$.
And also $(\tilde{m}_i)\tilde{M}_i = (\inf)\sup\{\sqrt{f(x)} : x \in [x_i, x_{i+1}]\} = (\sqrt{m_i})\sqrt{M_i}$.

$$\begin{aligned} U(\sqrt{f},\mathcal{P}) - L(\sqrt{f},\mathcal{P}) &= \sum_{i=1}^{n} (\tilde{M}_i - \tilde{m}_i)\Delta x_i \\ &= \sum_{i=1}^{n} (\sqrt{M_i} - \sqrt{m_i})\Delta x_i. \end{aligned}$$

We split the sum into two parts.

$$\begin{aligned} U(\sqrt{f},\mathcal{P}) - L(\sqrt{f},\mathcal{P}) &= \left(\sum_{M_i \geq \epsilon^2} + \sum_{M_i < \epsilon^2} \right) (\sqrt{M_i} - \sqrt{m_i})\Delta x_i \\ &\leq \frac{1}{\epsilon} \sum_{M_i \geq \epsilon^2} (M_i - m_i)\Delta x_i + \sum_{M_i < \epsilon^2} (\sqrt{M_i} - \sqrt{m_i})\Delta x_i \\ &\leq \frac{1}{\epsilon} \cdot \sum_{i=1}^{n} (M_i - m_i)\Delta x_i + \epsilon \cdot \sum_{i=1}^{n} \Delta x_i \\ &\leq \epsilon + \epsilon \cdot (b-a) = \epsilon(b-a+1). \end{aligned}$$

To conclude, $\forall\, \epsilon > 0$, $\exists$ partition $\mathcal{P}$ on $[a,b]$ such that

$$U(\sqrt{f},\mathcal{P}) - L(\sqrt{f},\mathcal{P}) < \epsilon(b-a+1).$$

$\square$

5. Suppose f is Riemann integrable on $[0, 1]$, find $\lim_{n\to\infty} \int_0^1 x^n f(x)\, dx$

Proof. The limit is zero. Since f is Riemann integrable, f is bounded. Let $M > 0$ such that $|f(x)| \leq M$ on $[0, 1]$.

$$\left| \int_0^1 x^n f(x)\, dx \right| \leq M \int_0^1 x^n\, dx = \frac{M}{n+1} \to 0.$$

□